XXVIII. *On the Summation of Series, whose general Term is a determinate Function of z the Distance from the first Term of the Series. By* Edward Waring, *M. D.* Lucasian *Professor of the Mathematics at* Cambridge, *and Fellow of the Societies of* London *and* Bononia.

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PROBLEM.

THE sum S *being given, to find a series of which it is the sum.*

1. Reduce the sum S into a converging series, proceeding according to the dimensions of any small quantities, and it is done. For example: let any algebraical function S of an unknown or small quantity x be assumed, reduce it into a converging series proceeding according to the dimensions of x, and there results a series whose sum is S. 2. Let A, B, C, &c. be algebraical functions of x; reduce the $\int A\dot{x}$, $\int B\dot{x}$, $\int C\dot{x}$, &c. into a converging series, proceeding according to the dimensions of x, and the problem is done.

It is always necessary to find the values of x, between which the abovementioned serieses converge. Reduce the algebraical function S in the first example, and the algebraical functions A, B, C, &c. in the second into their lowest terms; and in such a manner, that the quantities contained in the numerator and denominator may have no denominator: make the deno-

 minator

minator in the first example, and the denominator in the second, and every distinct irrational quantity contained in them respectively $=0$; and also every distinct irrational quantity contained in the numerators $=0$. Suppose α the least root affirmative or negative (but not $=0$) of the abovementioned resulting equations; then a series ascending according to the dimensions of x will always converge, if the value of x is contained between α and $-\alpha$; but if x be greater than α or $-\alpha$, the abovementioned series will diverge. Let π be the greatest root of the abovementioned resulting equations; then a series descending according to the reciprocal dimensions of x will converge, if x be greater than $\pm\pi$; but, if less, not. When impossible roots $a \pm b\sqrt{-1}$ are contained in the equations, an ascending series will converge, if x be less than the least root $\pm\alpha$, and $\pm(a-b)$ and $\pm(a+b)$; or more generally, if x be less than the least root $\pm\alpha$, and x^{n+1} at an infinite distance n, be infinitely less than

$$\frac{2a^n - 2 \cdot n \cdot \frac{n-1}{2} a^{n-2} b^2 + 2 \cdot n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} a^{n-4} b^4 - \&c.}{(a^2+b^2)^n}:$$

a descending series will always converge, when x is greater than the greatest root of the resulting equations; and x^{n-1}, when n is infinite, is infinitely greater than $(a+b)^n$ and $(a-b)^n$; or more generally than $2a^n - 2n \cdot \frac{n-1}{2} a^{n-2} b^2 + 2n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} a^{n-4} b^4 - \&c.$

This follows from Caput 3. of the Meditationes Algebraicæ.

Cor. It appears from hence, that, if the ascending series converges, the descending will diverge; and, *vice versâ*, if the descending converges, the ascending will diverge, unless all the roots of the above-mentioned resulting equations may be deemed of of equal magnitude, as $+\alpha$ and α, $\alpha\sqrt[n]{-1}$, &c. and $x=\alpha$; in which case sometimes both serieses may become the same converging series, &c.

When x, in the preceding cases, is equal to the least or greatest root, the series will sometimes converge, and sometimes not, as is shewn in the above-mentioned chapter. Whether the sum of a series, whose general term is given, can be found or not, will in many cases appear from the law of the multinomial and other more general serieses.

2. There are serieses which always converge, whatever may be the value of x; as, for example, the series $1+\frac{1}{2\cdot3}x+\frac{1}{2\cdot3\cdot4\cdot5}x^2+$&c. or $1+\frac{x}{2^2}+\frac{x^2}{3^3}+\frac{x^3}{4^4}+$&c. &c. always converge, whatever may be the value of x; but it may be observed, that these serieses never arise from the expansion of algebraical functions of x, or the before-mentioned fluents; but, in a few cases, they may from fluxional equations. There are also serieses which never converge as $1+1\cdot2x+1\cdot2\cdot3x^2+1\cdot2\cdot3\cdot4x^3+$&c. to which the preceding remark may be applied.

3. In the year 1757 some papers, which contained the first edition of my Meditationes Algebraicæ, were sent to the Royal Society, in which was contained the following rule, *viz.* let S be a given function of the quantity x, which expand into a series $(a+bx^m+cx^{2m}+\text{\&c.})$ proceeding according to the dimensions of x; in the quantity S, for x^m write αx^m, βx^m, γx^m, &c. where α, β, γ, &c. are roots of the equation $x^n-1=0$; and let the resulting quantities be A, B, C, D, &c. then will $\frac{A+B+C+D+\text{\&c.}}{n}$ be equal to the sum of the first, $2n+1$, $3n+1$, &c. terms *in infinitum*. This method, in the preface to the

last edition of the Meditationes Algebraicæ, is rendered more correct and general.

4. Let the sum of a series required be expressed by a function of a quantity z, the distance from the first term of the series, then will the general term be the difference between the two successive sums generally expressed.

5. Let the general term be an algebraical function of z: 1st, let it be $\frac{az^m + bz^{m-1} + cz^{m-2} + \&c.}{z+e . z+e+1 . z+e+2 \ldots z+e+n-1} = T$, where m and n are whole numbers; and m (if the sum of an infinite series of terms is required) less than n by two or more: then the general term $\frac{az^m + bz^{m-1} + \&c.}{z+e . z+e+1 . z+e+2 \ldots z+e+n-1} =$

$$\frac{\gamma}{z+e . z+e+1} + \frac{\delta}{z+e . z+e+1 . z+e+2} +$$

$$\frac{\varepsilon}{z+e . z+e+1 . z+e+2 . z+e+3} + \&c. \ldots \frac{\theta}{z+e . z+e+1 \ldots z+e+n-1},$$

whence if

$z+e+2 . z+e+3 . z+e+4 \ldots z+e+n-1 = z^{n-2} + Az^{n-3} + Bz^{n-4} + \&c.$; $z+e+3 . z+e+4 . z+e+5 \ldots z+e+n-1 = z^{n-3} + A'z^{n-4} + B'z^{n-5} + \&c.$; $z+e+4 . z+e+5 \ldots z+e+n-1 = z^{n-4} + A''z^{n-5} + B''z^{n-6} + \&c.$ and so on; then, if $m = n-2$, will $\gamma = a$, $\delta = b - \gamma A$, $\varepsilon = c - \delta A' - \gamma B$, $\zeta = d - \varepsilon A'' - \delta B' - \gamma C$, &c.; whence the integral *in infinitum*, or sum of the infinite series, will be $\frac{\gamma}{z+e} + \frac{\delta}{2 . z+e . z+e+1} +$

$$\frac{\varepsilon}{3 . z+e . z+e+1 . z+e+2} + \&c.$$

The reduction of the general term T into quantities of the before given formulæ was published in the Meditationes, printed in the year 1774. It was before reduced into formulæ of the same kind nearly by Mr. NICHOLE in the Paris Acts.

2d, Let the general term be $T' =$

$$\frac{az^b + bz^{b-1} + cz^{b-2} + \&c.}{z+e . z+e+1 . z+e+2 \ldots z+e+n-1 \times z+f . z+f+1 \ldots z+f+m-1 \times z+g . z+g+1 \ldots z+g+l-1 \times \&c}$$

where

where h is a whole number lefs than $n+m+l+$ &c. (if it be greater, then the fraction can eafily be reduced into a rational quantity $az^{h-n-m-l-\&c.}+$ &c. and a fraction of the before-mentioned kind); then will $T' = \left(\frac{\alpha}{z+e} + \frac{\alpha'}{z+f} + \frac{\alpha''}{z+g} + \&c.\right) + \left(\frac{\beta}{z+e \,.\, z+e+1} + \frac{\beta'}{z+f \,.\, z+f+1} + \frac{\beta''}{z+g \,.\, z+g+1} + \&c.\right) + \left(\frac{\gamma}{z+e \,.\, z+e+1 \,.\, z+e+2} + \frac{\gamma'}{z+f \,.\, z+f+1 \,.\, z+f+2} + \frac{\gamma''}{z+g \,.\, z+g+1 \,.\, z+g+2} + \&c.\right) \ldots \left(\frac{X}{z+e \,.\, z+e+1 \,..\, z+e+n-1} + \frac{X'}{z+f \,.\, z+f+1 \,..\, z+f+m-1} + \frac{X''}{z+g \,.\, z+g+1 \,..\, z+g+l-1} + \&c.\right)$; whence its integral *in infinitum*, that is, the fum of the infinite feries can be found when $\alpha=0$, $\alpha'=0$, $\alpha''=0$, &c.; and confequently h not greater than $n+m+l+$ &c. -2; otherwife not. If h is not greater than $n+m+l+$ &c. -2, then will $\alpha+\alpha'+\alpha''+$ &c. $=0$, for elfe the fum would be infinite.

Let the number of quantities (e, f, g, &c.) be r, then from r independent integrals of a feries, whofe term is T'; or from $(r-1)$ independent fums of infinite feriefes, whofe term is T'; that is, where h is not greater than $n+m+l+$ &c. -2; can be deduced the fum of all infinite feriefes of the before-mentioned formulæ, whofe general term is T'.

If any factors are deficient in the denominator, as fuppofe the term to be $z+e \times z+e+3 \times z+e+n-1$; multiply the numerator and denominator by the deficient factors, *viz.* by $z+e+1 \,.\, z+e+2 \times z+e+4 \,.\, z+e+5 \,..\, z+e+n-2$, and it acquires the preceding formula; and fo in the following examples.

3d, Let the denominator be $\overline{x+e}^{\pi} \times \overline{x+e+1}^{\pi} \times \overline{x+e+2}^{\pi} \ldots \overline{x+e+n-1}^{\pi} \times \overline{x+e+\mu}^{\pi'} \times \overline{x+e+\mu+1}^{\pi'} \times$

$x+e$

$\overline{x+e+\mu+2}^{\pi'} \times$ &c. $\times \overline{x+f}^{\varrho} \times \overline{x+f+1}^{\varrho} \times \overline{x+f+2}^{\varrho} \ldots \times \overline{x+f+m-1}^{\varrho} \times$ &c. $=$ D, where π, π', ϱ, &c.; μ, &c. are whole numbers; and the general term is $\frac{az^b+bz^{b-1}+cz^{b-2}+\&c.}{D}$ $=$ T″; then, if the dimenſions of z in the numerator be leſs than its dimenſions in the denominator, will T″ $=$

$$\left(\frac{\alpha}{z+e}+\frac{\alpha'}{(z+e)^2}+\frac{\alpha''}{(z+e)^3}\ldots\frac{\alpha'^{\pi+\pi'-1}}{(z+e)^{\pi+\pi'}}+\frac{\beta}{z+f}+\frac{\beta'}{(z+f)^2}+\frac{\beta''}{(z+f)^3}\ldots\frac{\beta'^{\varrho-1}}{(z+f)^{\varrho}}+\&c.\right)+\left(\frac{\gamma}{z+e\,.\,z+e+1}+\frac{\theta}{z+f\,.\,z+f+1}+\&c.\right)+\&c.;$$

and in general there will be included all terms of the formulæ,

$$\frac{A\left(\overline{z+e+i}^{p}-\overline{z+e}^{p}\right)}{(z+e)^p\,.\,(z+e+1)^p\ldots(z+e+i)^p},$$

$$\frac{B\left((z+f+i')^{p'}-(z+f)^{p'}\right)}{(z+f)^{p'}\,.\,(z+f+1)^{p'}\ldots(z+f+i')^{p'}},\ \&c.$$

$$\frac{C\left((z+e+\mu+i'')^{p''}-(z+e+\mu)^{p''}\right)}{(z+e+\mu)^{p''}\,.\,(z+e+\mu+1)^{p''}\ldots(z+e+\mu+i'')^{p''}},\ \&c.$$

where A, B, C, &c. α, α', &c. β, β', &c. γ, θ, &c. denote invariable quantities; and p, p', p'', &c. are whole numbers not greater than π, ϱ, π', &c. reſpectively; and i, i', i'', &c. are whole numbers not greater than $n-1$, $m-1$, &c.

If all the quantities α, α', α'', &c. β, β', β'', &c. &c. are $=0$, the ſum of the ſeries can be expreſſed in finite terms of the quantity z, otherwiſe not; and alſo if b be leſs than the dimenſions of z in the denominator by two or more, then will $\alpha+\beta+$&c.$=0$, otherwiſe the ſum would be infinite.

From $\pi+\pi'+\varrho+$&c.-1 independent ſums of infinite ſerieſes of this kind can be deduced the ſums of all infinite ſerieſes of the ſame kind.

2 This

This method may be extended to infinite ſeries, in which exponentials as e^z are contained, which will eaſily be ſeen from ſome ſubſequent propoſitions; but in my opinion the ſubſequent method of finding the ſum of ſerieſes is to be preferred to the preceding one, both for its generality and facility.

6. 1. Let the general term be $(az^b+bz^{b-1}+cz^{b-2}+\&c.)\times(z+e)^{-1}.(z+e+1)^{-1}.(z+e+2)^{-1}\ldots(z+e+n-1)^{-1}$; where b is a whole number leſs than n by two or more, when the ſum of an infinite ſeries is required.

Aſſume for the ſum the quantity $(z+e)^{-1}.(z+e+1)^{-1}.(z+e+2)^{-1}\ldots(z+e+n-2)^{-1}\times(\alpha z^{b'}+\beta z^{b'-1}+\gamma z^{b'-2}+\&c.)$; find the difference between this ſum and its ſucceſſive one $(z+e+1)^{-1}.(z+e+2)^{-1}.(z+e+3)^{-1}\ldots(z+e+n-1)^{-1}\times(\alpha\overline{z+1}^{b'}+\beta\overline{z+1}^{b'-1}+\&c.)$, which will be $-(z+e)^{-1}.(z+e+1)^{-1}.(z+e+2)^{-1}\ldots(z+e+n-1)^{-1}\times(\overline{z+e}\times(\alpha\overline{z+1}^{b'}+\beta\overline{z+1}^{b'-1}+\&c.)-\overline{z+e+n-1}\times(\alpha z^{b'}+\beta z^{b'-1}+\&c.)=\overline{b'-n+1}\alpha z^{b'}+\&c.)$; then make the terms of this difference equal to the correſpondent terms of the given quantity $az^b+bz^{b-1}+\&c.$ and there reſult $b'=b$, $-\overline{b-n+1}\times\alpha=a$, and conſequently $\alpha=\frac{-a}{b-n+1}$, &c.

2. Let the general term be $(z+e)^{-1}.(z+e+1)^{-1}.(z+e+2)^{-1}\ldots(z+e+n-1)^{-1}\times(z+f)^{-1}.(z+f+1)^{-1}.(z+f+2)^{-1}\ldots(z+f+m-1)^{-1}\times(az^b+bz^{b-1}+cz^{b-2}+\&c.)$. Aſſume the quantity $(z+e)^{-1}.(z+e+1)^{-1}\ldots(z+e+n-2)^{-1}\times(z+f)^{-1}.(z+f+1)^{-1}.(z+f+2)^{-1}..(z+f+m-2)^{-1}\times(\alpha z^{b'}+\beta z^{b'-1}+\gamma z^{b'-2}+\&c.)$ for the ſum of the ſeries ſought; and thence deduce the general term, which ſuppoſe equal to the given general term, and from equating their correſponding parts eaſily can be deduced the index b' and co-efficients α, β, γ, &c. and conſequently the ſum of the ſeries ſought.

3. Let

3. Let the general term reduced to its lowest dimensions be $\overline{z+e}^{\pi} \times \overline{z+e+1}^{\pi} \ldots \overline{z+e+n-1}^{\pi} \times \overline{rz+f}^{-\varrho} \times \overline{rz+f+r}^{-\varrho} \times \overline{rz+f+2r}^{-\varrho} \ldots \overline{rz+f+m-1r}^{-\varrho} \times \overline{z+g}^{-\sigma} \times \overline{z+g+1}^{-\sigma} \times \ldots \overline{z+g+l-1}^{-\sigma} \times$ &c. $\times (az^{b} + bz^{b-1} + cz^{b-2} + \&c.)$. If it be required to reduce the term $\overline{rz+f}^{-\varrho}$, &c. to a conformity with the rest, for $\overline{rz+f}^{-\varrho}$, &c. substitute $\overline{z+\frac{f}{r}}^{-\varrho} \times r^{-\varrho}$, &c. and it is done. Assume for the integral or sum the quantity $S = \overline{z+e}^{\pi} . \overline{z+e+1}^{\pi} .. \overline{z+e+n-2}^{\pi} \times \overline{rz+f}^{-\varrho} . \overline{rz+r+f}^{-\varrho} .. \overline{rz+m-2r+f}^{-\varrho} \times \overline{z+g}^{-\sigma} \times \overline{z+g+1}^{-\sigma} \ldots \overline{z+g+l-2}^{-\sigma} \times$ &c. $\times (\alpha z^{b'} + \beta z^{b'-1} + \&c.) = S$, find its successive sum by writing $z+1$ for z in the sum S, and let the quantity resulting be S′; then will the general term be S − S′, which equate to the given general term, that is, their correspondent quantities; and thence may be deduced the index b' and co-efficients α, β, &c.; and consequently the sum sought. If the series does not terminate, then the sum will be expressed by a series proceeding *in infinitum*, according to the reciprocal dimensions of z.

From $\pi + \varrho + \sigma +$ &c. $- 1$ independent integrals of the above-mentioned kind can be deduced the integrals of all quantities of the same kind; that is, where h is any whole affirmative number whatever, and the co-efficients a, b, c, &c. are any how varied.

If any factor $z+g$ in the denominator, &c. has no other $z+g+l-1$, which differs from it by a whole number $l-1$; or the factor $rz+f$ has no correspondent factor $rz+f+mr$, where m is a whole number; then the integral of the above-mentioned series cannot be expressed in finite terms of the quantity z. In like manner, if the dimensions of z in the numerator are less than

than its dimenſions in the denominator by unity, then the integral of the general term cannot be expreſſed by a finite algebraical function of z. If the number of terms to be added be infinite, it is well known that the ſum in this caſe will be infinite.

It may be obſerved, that in finding the ſum of a ſeries, whoſe general term is given, all common diviſors of the numerator and denominator muſt be rejected, otherwiſe ſerieſes may appear difficult to be ſummed, which are very eaſy: for example, let the ſeries be $\frac{1}{1.2.3.4.5}+\frac{4}{4.5.6.7.8}+\frac{9}{7.8.9.10.11}$ $+\&c.=\frac{1}{3}\left(\frac{1}{1.2.4.5}+\frac{2}{4.5.7.8}+\frac{3}{7.8.10.11}+\&c.\right)$, whoſe general term is $\frac{z+1}{3z+1\,.\,3z+4\times 3z+2\,.\,3z+5}$; and by aſſuming, as is before taught, $\overline{3z+1}^{-1}\times\overline{3z+2}^{-1}\times\alpha$ for the ſum ſought; and finding its general term $\overline{3z+1}^{-1}\times\overline{3z+4}^{-1}\times\overline{3z+2}^{-1}\times$ $\overline{3z+5}^{-1}\times 18\,\overline{z+1}\times\alpha$, which equating to the general term given, there reſults $18\alpha=1$, and the ſum ſought $=\frac{1}{18}\times$ $\frac{1}{3z+1\,.\,3z+2}$.

Ex. 2. Let the ſeries be $\frac{1}{1.2.3.4.5}+\frac{14}{5.6.7.8.9}+$ $\frac{55}{9.10.11.12.13}+\frac{140}{13.14.15.16.17}+\&c.=\frac{1}{24}\left(\frac{1}{1.5}+\frac{1}{5.9}+\right.$ $\left.\frac{1}{9.13}+\frac{1}{13.17}+\&c.\right)$, of which the general term is $\frac{1}{24}\times$ $\frac{1}{4z+1\,.\,4z+5}$; and conſequently the ſum deduced is $\frac{1}{24}\times$ $\frac{1}{4}\times\frac{1}{4z+1}$.

Theſe are ſerieſes given by Mr. De Moivre, and eſteemed by Dr. Taylor *altioris indaginis.*

 Some

Some other writers have made ſome ſerieſes to appear more difficult to be ſummed, by not reducing them to their loweſt terms.

7. Having given the principles of a general method of finding the ſum of a ſeries, when its general term can be expreſſed by algebraical, and not exponential, functions of z, the diſtance from the firſt term of the ſeries; it remains to perform the ſame when exponentials are included.

1. Let S the ſum be any algebraical function of z multiplied into $e^z = x^z$; then will the general term be $Se^z - eS'e^z = (S - eS')\, e^z$; whence, from the general term Te^z being given, aſſume quantities in the ſame manner (with the ſame denominator, &c.) as when no exponential was involved, which multiplied into e^z, ſuppoſe to be the ſum; from the ſum find its general term, and equate it to the given one by equating their correſpondent co-efficients, and it is done.

Ex. Let the general term be $\frac{z+2}{2z+1 \cdot 2z+3} \times e^{z+1}$: aſſume for the ſum ſought $\frac{\alpha}{2z+1} \times e^{z+1}$, whence the general term is $\left(\frac{\alpha}{2z+1} - \frac{\alpha e}{2z+3}\right) e^{z+1} = \frac{2\alpha(1-e)z + 3\alpha - \alpha e}{2z+1 \cdot 2z+3} \times e^{z+1}$; equate it to the given term, and there reſults $2\alpha(1-e) = 1$ and $3\alpha - \alpha e = 2$, and conſequently $e = \frac{1}{3}$ and $\alpha = \frac{3}{4}$, if the ſeries can be ſummed.

The ſame obſervation, *viz.* that if any factor in the denominator or irrational quantity have no other correſpondent to it; for example, if the factor be $z+g$, and there is no correſpondent one $x+g+n$, where n is a whole number, then its integral cannot be expreſſed by a finite algebraical function of z.

In the ſame manner may the ſums be found, when the terms are exponentials of ſuperior orders; for the exponential, irra-

6 tional,

tional, &c. quantities in the denominators of the ſums may be eaſily deduced from the preceding principles; and thence, by proceeding as is before taught, the ſum required.

The principles of all theſe caſes have been given in the Meditationes.

8. Mr. JAMES BERNOULLI found ſummable ſerieſes by aſſuming a ſeries V, whoſe terms at an infinite diſtance are infinitely little, and ſubtracting the ſeries diminiſhed by any number (l) of terms from the ſeries itſelf, &c.

It is obſerved in the Meditationes, that if T (m), T ($m+n$), T ($m+n+n'$), T ($m+n+n'+n''$), &c. be the terms at m, $m+n$, $m+n+n'$, $m+n+n'+n''$, &c. diſtances from the firſt, and aT (m) + bT ($m+n$) + cT ($m+n+n'$) + dT ($m+n+n'+n''$) + &c. be the general term, it will be ſummable, when $a+b+c+d+$ &c. = 0; the ſum of the ſeries will be a(T(m) + T ($m+1$) + T ($m+2$) + . . . + T ($m+n+n'+n''+$ &c. -1)) + b(T ($m+n$) + T ($m+n+1$) + T ($m+n+2$) + . . . + T ($m+n+n'+n''+$ &c. -1)) + c(T($m+n+n'$) + T($m+n+n'+1$) + . . (T($m+n+n'+n''+$ &c. -1)) + &c. = H. If the ſum $a+b+c+d+$ &c. be not = 0, and the ſeries T(m) + T($m+1$) + T($m+2$) + &c. *in infinitum* be a converging one = S, then will the ſum of the reſulting ſeries be ($a+b+c+d+$ &c.) S − ($b+c+d+$ &c.) (T^{m} . . . + T^{m+n-1}) − ($c+d+$ &c.) (T^{m+n} . . . $T^{m+n+n'-1}$) − ($d+$ &c.) ($T^{m+n+n'}$ + . . . $T^{m+n+n'+n''-1}$) − &c.

8. 2. Let the ſeries V conſiſt of terms, which have only one factor in the denominator, and its numerator = 1; that is, let the general term be $\frac{1}{rz+e}$, and the ſeries conſequently $\frac{1}{e}+\frac{1}{r+e}+\frac{1}{2r+e}+$ &c. = V; from the before-mentioned addition or ſubtraction there follows $\frac{a}{rz+e}+\frac{b}{rz+r+e}+\frac{c}{rz+2r+e}+$ &c. =

 $az^{m}+$

$\frac{\alpha z^{m} + \beta z^{m-1} + \gamma z^{m-2} + \&c.}{rz+e \,.\, rz+r+e \,.\, rz+2r+e \,.\, \&c.}$; where m is not greater than the number (N) of factors in the denominator diminished by unity. From α, β, γ, &c. r and e being given, easily can be acquired by simple equations, or known theorems, the required co-efficients a, b, c, &c. If $m = \mathrm{N} - 1$ and α and $a+b+c+d+\&c. = 0$, then the sum of the series resulting will be finite.

8. 3. If the terms of the series assumed $\frac{1}{e} - \frac{1}{r+e} + \frac{1}{2r+e} - \frac{1}{3r+e} + \&c.$ be alternately affirmative and negative; then by the preceding case find $\frac{\alpha z^{m} + \beta z^{m-1} + \gamma z^{m-2} + \&c.}{rz+e \,.\, rz+r+e \,.\, rz+2r+e + \&c.} = \frac{a}{rz+e} + \frac{b}{rz+r+e} + \frac{c}{rz+2r+e} + \&c.$ Where the terms of the resulting series are alternately affirmative and negative, let the two subsequent terms be supposed $\frac{\alpha z^{m} + \beta z^{m-1} + \gamma z^{m-2} + \&c.}{rz+e \,.\, rz+r+e \ldots rz+\overline{n-1}r+e} = \frac{a}{rz+e} + \frac{b}{rz+r+e} + \&c.$ and $\frac{\alpha \overline{z+1}^{m} + \beta \overline{z+1}^{m-1} + \gamma \overline{z+1}^{m-2} + \&c.}{rz+r+e \,.\, rz+2r+e \ldots rz+nr+e} = \frac{a}{rz+r+e} + \frac{b}{rz+2r+e} + \&c.$ of which the one is affirmative and the other negative: reduce the resulting series to an affirmative one by subtracting the subsequent term from its preceding, and it becomes $\frac{(rz+nr+e)\,(\alpha z^{m} + \beta z^{m-1} + \&c.) - (rz+e)\,(\alpha \overline{z+1}^{m} + \beta \overline{z+1}^{m-1} + \&c.)}{rz+e \,.\, rz+r+e \,.\, rz+2r+e \ldots rz+rn+e} = \frac{\overline{n-mr}\,\alpha z^{m} + \&c.}{rz+e \,.\, rz+r+e \ldots rz+rn+e} = \frac{a}{rz+e} + \frac{b-a}{rz+r+e} + \&c.$ In this case, since two terms are added into one, the distance from the first term of the series will be $\frac{z}{2}$, which suppose $= w$; and write $2w$ for z in the above-mentioned term, and there results $\frac{\overline{n-mr}\,\alpha z^{m} + \&c.}{rz+e \,.\, rz+r+e \ldots rz+nr+e} = \frac{\overline{n-mr}\,\alpha \times 2^{m} w^{m} + \&c.}{2rw+e \,.\, 2rw+r+e \ldots 2rw+nr+e} =$

$\frac{a}{2rw+e}$

$\frac{a}{2rw+e}+\frac{b-a}{2rw+r+e}+$ &c.; whence the ſum of any ſeries, whoſe general term is $\frac{a'w^m+b'w^{m-1}+\&c.}{2rw+e\,.\,2rw+r+e\ldots 2rw+nr+e}$, where m is a whole number leſs than n by two or more, and w the diſtance from the firſt term of the ſeries can be found from the ſum of the ſeries $\frac{1}{e}-\frac{1}{r+e}+\frac{1}{2r+e}-\frac{1}{3r+e}+$ &c.

9. Let there be two ſerieſes $\frac{1}{e}+\frac{1}{e+r}+\frac{1}{e+2r}+$ &c. $=S$ and $\frac{1}{f}+\frac{1}{f+r}+\frac{1}{f+2r}+\frac{1}{f+3r}+$ &c. $=S'$, whoſe general terms are reſpectively $+\frac{1}{e+rz}$ and $+\frac{1}{f+rz}$; then from the ſum of theſe two ſerieſes can be collected the ſum of any ſeries, whoſe general term is

$$\frac{\alpha z^m+\beta z^{m-1}+\&c.}{rz+e\,.\,rz+e+r\,.\,rz+e+2r\ldots rz+\overline{n-1}r+e\times rz+f\,.\,rz+r+f\ldots rz+f+\overline{m-1}r}$$
$$=\frac{a}{rz+e}+\frac{b}{rz+e+r}+\frac{c}{rz+e+2r}\ldots+\frac{\lambda}{rz+\overline{n-1}r+e}+\frac{a'}{rz+f}+\frac{b'}{rz+r+f}$$
$$+\frac{c'}{rz+2r+f}\ldots+\frac{\mu'}{rz+\overline{m-1}r+f};$$ where $e-f$ is not a whole number. Let $a+b+c\ldots+\lambda=0$, and $a'+b'+c'\ldots+\mu'=0$, then the ſum will be $a\left(\frac{1}{rz+e}+\frac{1}{rz+r+e}\ldots+\frac{1}{rz+\overline{n-2}r+e}\right)+b\left(\frac{1}{rz+e+r}+\frac{1}{rz+e+2r}\ldots+\frac{1}{rz+\overline{n-2}r+e}\right)+c\left(\frac{1}{rz+e+2r}+\frac{1}{rz+e+3r}+\ldots+\frac{1}{rz+\overline{n-2}r+e}\right)+$ &c. $+a'\left(\frac{1}{rz+f}+\frac{1}{rz+r+f}\ldots\frac{1}{rz+\overline{m-2}r+f}\right)+b'\left(\frac{1}{rz+r+f}+\ldots+\frac{1}{rz+\overline{m-2}r+f}\right)+$ &c.

2. If the ſerieſes are $\frac{1}{e}-\frac{1}{e+r}+\frac{1}{e+2r}-$ &c. and $\frac{1}{f}-\frac{1}{f+r}+\frac{1}{f+2r}-$ &c.; then from the ſum of theſe two ſerieſes can be collected by the principles given above the ſum of any ſeries, whoſe general term is

$$\frac{\alpha z^{m} + \beta z^{m-1} + \gamma z^{m-2} + \&c.}{2rz+e \cdot 2rz+r+e \cdot 2rz+2r+e \ldots 2rz+\overline{n-1}r+e \times 2rz+f \cdot 2rz+r+f \cdot 2rz+2r+f \ldots 2rz+\overline{m-1}r+f}$$

The same principle may be applied to find the sum of any series of the abovementioned sort, in whose denominator are contained other factors, $rz+g$, $rz+g+r$, &c. &c.; or $2rz+g$, $2rz+g+r$, $2rz+g+2r$, &c. Like propositions may be deduced from serieses, in which r and r', &c. and the factors $rz+e$ and $r'z+g$, &c. denote different quantities.

10. An apparently more general method may be given from assuming a series or serieses as before; and adding every two, three, four, &c. (n) successive terms together for terms of a new series beginning from the first, second, third, &c. n^{th} term; and in general adding together two, three, &c. n successive general terms; and in their sum writing for z the distance from the first term of the series $2z+a$, $3z+a$, &c. $nz+a$; there will result the general term of a series not to be found from the above-mentioned addition.

Ex. Let the series assumed be $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+$ &c. *in infinitum*, of which the general term beginning from the first is $\frac{1}{z+1}$; add three successive general terms $\frac{1}{z+1}+\frac{1}{z+2}+\frac{1}{z+3}$ $=\frac{3z^2+12z+11}{z+1 \cdot z+2 \cdot z+3}$; in this term for z write $3z$, and there results $\frac{27z^2+36z+11}{3z+1 \cdot 3z+2 \cdot 3z+3}$. In the same manner, if the beginning is instituted from the second or third term of the given series, the terms resulting will be $\frac{3z^2+18z+26}{z+2 \cdot z+3 \cdot z+4}$ and $\frac{3z^2+24z+47}{z+3 \cdot z+4 \cdot z+5}$. In these terms for z write $3z$, and there result $\frac{27z^2+54z+26}{3z+2 \cdot 3z+3 \cdot 3z+4}$ and $\frac{27z^2+72z+47}{3z+3 \cdot 3z+4 \cdot 3z+5}$.

If

If the terms of the given ſeries are alternately affirmative and negative, the terms of the reſulting ſeries will be alternately affirmative and negative, if n be an odd number; otherwiſe its terms will be all affirmative. The ſum of this ſeries will be finite or infinite, as the ſum of the ſeries $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c.$ is finite or infinite; but from it, by the preceding method of addition or ſubtraction of Mr. BERNOULLI's, or a like method applied to more ſerieſes, may be found the ſums of different finite ſerieſes.

It may be obſerved, that from Mr. BERNOULLI's addition or ſubtraction can never be deduced the ſerieſes which ariſe from this method; for, by his method, the denominator can never have any factors but what are contained in the denominators of the given ſeries, *viz.* (in the ſeries $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \&c.$), $z + l$, where l is a whole number; but by this method are introduced into the denominator the factors $2z + l$, $3z + l$, &c. and $nz + l$, or which may be reduced to the ſame $\left(z + \frac{l}{n}\right) \times n$.

If n ſucceſſive general terms of the ſerieſes ariſing from Mr. BERNOULLI's addition or ſubtraction be added together, and in the quantity thence ariſing for z the diſtance from the firſt term of the ſeries be ſubſtituted nz, there will be produced ſerieſes of the above-mentioned formula.

11. Multiply two converging ſerieſes $a + bx + cx^2 + dx^3 + \&c. = S$ and $\alpha + \beta x + \gamma x^2 + \&c. = V$, or find any rational and integral function of them, and the ſeries reſulting will be finite and $= S \times V$, &c. Let $\alpha + \beta x + \gamma x^2 + \&c.\ x^m = V$ be finite, and the reſulting ſeries will be finite and $= S \times V$, &c. If S be a ſeries converging or not, whoſe ultimate terms are leſs than any finite quantity, then will the ſeries $(a + bx + cx^2 + \&c.) \times (\alpha + \beta x + \gamma x^2 + \&c.\ x^m) = V \times S$ be a converging one, if $\alpha + \beta x + \gamma x^2 + \ldots \&c.\ x^m = 0$; which caſe was given by Mr. DE MOIVRE.

Mr.

Mr. BERNOULLI's addition, &c. can be applied to serieses of this kind. For example, let the given series be $\frac{1}{e}+\frac{1}{e+1}x+\frac{1}{e+2}x^2+$ &c. $=$ S. From this series subtract the same series diminished by m terms, *viz.* $\frac{1}{e+m}x^m+\frac{1}{e+m+1}x^{m+1}+\frac{1}{e+m+2}x^{m+2}$ + &c. and there remains $\frac{e+m-ex^m}{e\,.\,m+e}+\frac{e+m+1-\overline{e+1}x^m}{e+1\,.\,e+m+1}x+\frac{e+m+2-\overline{e+2}x^m}{e+2\,.\,e+m+2}x^2+\frac{e+m+3-\overline{e+3}x^m}{e+3\,.\,e+m+3}\times x^3+$ &c.; for x^m write A, then will the series become $\frac{m-e\mathrm{A}}{e\,.\,m+e}+\frac{e+m+1-\overline{e+1}\mathrm{A}}{e+1\,.\,e+m+1}x+\frac{e+m+2-\overline{e+2}\mathrm{A}}{e+2\,.\,e+m+2}x^2+\frac{e+m+3-\overline{e+3}\mathrm{A}}{e+3\,.\,e+m+3}x^3+$ &c. $=\frac{1}{e}+\frac{1}{e+1}x+\frac{1}{e+2}x^2\,\ldots\,\frac{1}{e+m-1}x^{m-1}$.

Let the general term be $\frac{az^m+bz^{m-1}+cz^{m-2}+\&c.}{z+e\,.\,z+e+1\,.\,z+e+2\,.\,.\,z+e+n-1}\times x^z$ $=\left(\frac{\alpha}{z+e}+\frac{\beta}{z+e+1}+\frac{\gamma}{z+e+2}\,\ldots\,\frac{\mathrm{X}}{z+e+n-1}\right)x^z$. Suppose $\beta=\beta'x$, $\gamma=\gamma'x^2$, $\delta=\delta'x^3$, . . $\mathrm{X}=\mathrm{X}'x^{n-1}$; then will the sum of the above-mentioned series be $(\alpha+\beta'+\gamma'+\delta'+\&c.)\times \mathrm{S}-\frac{1}{e}(\beta'+\gamma'+\delta'+\&c.)-\frac{1}{e+1}(\gamma'+\delta'+\&c.)-\frac{1}{e+2}(\delta'+\&c.)-\&c.$

From the sum of the series $\frac{1}{e}-\frac{x}{e+1}+\frac{x^2}{e+2}-$ &c. by these and the principles before delivered can be deduced the sum of any series, whose general term is

$$\frac{az^m+bz^{m-1}+\&c.}{2z+e\,.\,2z+e+1\,.\,2z+e+2\,.\,2z+e+3\times\&c.}x^z.$$

In like manner from the sum of the serieses $\frac{x}{e}+\frac{x^2}{e+1}+\frac{x^3}{e+2}$ + &c. $\frac{x}{f}+\frac{x^2}{f+1}+\frac{x^3}{f+2}+$ &c. $\frac{x}{g}+\frac{x^2}{g+1}+\frac{x^3}{g+2}+$ &c. &c. can be deduced

deduced the sum of any series, whose general term is

$$\frac{az^m + bz^{m-1} + \&c.}{z+e \cdot z+e+1 \cdot z+e+2 \times \&c. \times z+f \cdot z+f+1 \cdot z+f+2 \cdot \&c. \times z+g \cdot z+g+1 \cdot \&c.} \times x^z.$$

And also from the sum of the serieses $\frac{1}{e} - \frac{1}{e+1}x + \frac{x^2}{e+2} - \&c.$ $\frac{1}{f} - \frac{x}{f+1} + \frac{x^2}{f+2} - \&c.$ $\frac{1}{g} - \frac{x}{g+1} + \frac{x^2}{g+2} - \&c.$ &c. can be deduced the sum of any series, whose general term is

$$\frac{az^m + bz^{m-1} + \&c.}{2z+e \cdot 2z+e+1 \cdot \&c. \times 2z+f \cdot 2z+f+1 \cdot \&c. \; 2z+g \cdot 2z+g+1 \cdot \&c.} \times x^z.$$

The method of adding more terms of a given series together, as before taught, may be applied to these and all other serieses. For example: let the given series be $1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \&c.$; add two terms constantly together, and it becomes $1 + \frac{1}{2}x + \&c. = \frac{2+x}{2} + \frac{4+3x}{3 \cdot 4}x^2 + \frac{6+5x}{5 \cdot 6}x^4 + \&c. = \frac{2+A}{2} + \frac{4+3A}{3 \cdot 4}x^2 + \frac{6+5A}{5 \cdot 6}x^4 + \&c.$ whence the general term is $\frac{2z+2+(2z+1)}{2z+2}$ $\frac{x}{2z+1}x^{2z}$. From the methods before given of addition, subtraction, and multiplication; and the serieses found by this method, can be derived serieses, whose sums are known.

12. Suppose a given series $ax^n + bx^{n \pm s} + cx^{n \pm 2s} + dx^{n \pm 3s} + \&c.$ whose sum p is either an algebraical, exponential, or fluential fluxion of x; multiply the equation $p = ax^n + bx^{n \pm s} + cx^{n \pm 2s} + dx^{n \pm 3s} + \&c.$ into $x^{\pm r - n}$, and there results $ax^{\pm r - n}p = ax^{\pm r} + bx^{\pm r \pm s} + cx^{\pm r \pm 2s} + \&c.$; find the fluxion of this equation, and there follows $\frac{1}{\dot{x}}$ multiplied into the fluxion of the quantity $(x^{\pm r - n}p)$ $= \pm rax^{\pm r - 1} + (\pm r \pm s)\, bx^{\pm r \pm s - 1} + (\pm r \pm 2s)\, cx^{\pm r \pm 2s - 1} + \&c.$ of which the general term is $(\pm r \pm zs) \times t$, where z denotes the distance from the first term of the series, and t is

is the term in the given ſeries, whoſe diſtance from the firſt is z. In the ſame manner may be deduced the ſum of a ſeries whoſe general term is $t' \times \overline{\pm r \pm zs} \times \overline{\pm r' \pm z \pm ns}$, or by repeated operations $t' \times \overline{ez^2 + fz + g}$, where t' is a term of the given equation, whoſe diſtance from the firſt term is z. And in general, from the ſum of a given ſeries, whoſe fluxion can be found, and whoſe general term is t', can be deduced by continued multiplication, and finding the fluxion, the ſum of a ſeries or quantity, of which the general term is At', where A is any function of the following kind $a'z^m + b'z^{m-1} + c'z^{m-2} + \&c.$ in which z denotes the diſtance from the firſt term of the ſeries, and m a whole number. It is to be obſerved, that if the given ſeries converges in a ratio, which is at leaſt equal to the ratio of the convergency of ſome geometrical ſeries, the reſulting equation will always converge. But if in a leſs ratio, then it will ſometimes converge, ſometimes not, according to the ratio which the ſucceſſive terms of the reſulting ſeries have to each other at an infinite diſtance.

Corollary. $\frac{p \cdot p+1 \cdot p+2 \cdot p+3 \cdot \cdot p+z}{r \cdot r+1 \cdot r+2 \cdot r+3 \cdot \cdot r+z} = \frac{p+z \cdot p+z-1 \cdot p+z-2 \cdot p+z-3 \cdot \cdot \cdot z+r+1}{r \cdot r+1 \cdot r+2 \cdot r+3 \cdot \cdot \cdot \cdot \cdot p-1}$, if $p-r$ be a whole affirmative number; but this latter quantity has the formula above-mentioned $az^m + bz^{m-2} + cz^{m-3} + \&c.$; and conſequently, if the ſum of the ſeries $a + bx^s + cx^{2s} + dx^{3s} + \&c. = p$ be known, by this method can be deduced the ſum of the ſeries $a + \frac{p}{r}bx^s + \frac{p \cdot p+1}{r \cdot r+1}cx^{2s} + \frac{p \cdot p+1 \cdot p+2}{r \cdot r+1 \cdot r+2}dx^{3s} + \&c.$

Ex. 1. Since $\overline{a+x}^{\frac{m}{n}} = a^{\frac{m}{n}}\left(1 + \frac{m}{n} \times \frac{x}{a} + \frac{m}{n} \times \frac{m-n}{2n}a^{-2}x^2 + \frac{m}{n} \cdot \frac{m-n}{2n} \cdot \frac{m-2n}{3n}a^{-3}x^3 + \&c.\right)$; multiply the ſucceſſive terms of this ſeries into

into the terms of the series 1, $\frac{p}{r}$, $\frac{p \cdot \overline{p+1}}{r \cdot \overline{r+1}}$, &c. and a series is deduced $a^{\frac{m}{n}} + \frac{p \cdot m}{r \cdot n} a^{\frac{m}{n}-1} x + \frac{p \cdot \overline{p+1} \times m \cdot \overline{m-n}}{r \cdot \overline{r+1} \cdot n \cdot 2n} x^{\frac{m}{n}-2} +$ &c. whose sum is known, if the sum of the series $= \overline{a+x}^{\frac{m}{n}}$ is known.

Ex. 2. If the series begins from the $l+1^{th}$ term of the above-mentioned binomial theorem $a^{\frac{m}{n}} + \frac{m}{n} a^{\frac{m}{n}-1} x +$ &c. *viz.* the series be $H \times \overline{1 + \frac{m-\overline{l+1}n}{\overline{l+2n}} \frac{x}{a} + \frac{m-\overline{l+2n}}{\overline{l+3n}} \frac{x^2}{a^2} + \frac{m-\overline{l+3n}}{\overline{l+4n}} \frac{x^3}{a^3}} +$ &c. of which let the respective terms be multiplied into 1, $\frac{p}{r}$, $\frac{p \cdot \overline{p+1}}{r \cdot \overline{r+1}}$, &c. there will result a series whose sum is known.

Ex. 3. From the rule first given by me for finding the sum of the terms at h distances from each other, the sum of the series $1 + \frac{m-\overline{l+1}n}{\overline{l+2} \cdot n} \times \frac{m-\overline{l+2n}}{\overline{l+3} \, n} \cdot \cdot \frac{m-\overline{l+hn}}{\overline{l+h+1n}} \times \frac{x^h}{a^h} + P \times \frac{m-\overline{l+h+1n}}{\overline{l+h+2n}} \times \frac{m-\overline{l+h+2n}}{\overline{l+h+3n}} \cdot \cdot \cdot \frac{m-\overline{l+2hn}}{\overline{l+2h+1n}} \frac{x^{2h}}{a^{2h}} +$ &c. where P denotes the co-efficient of the preceding term, can be deduced; and consequently the sum of the series deduced from multiplying the successive terms of this series into the quantities 1, $\frac{p}{r}$, $\frac{p \cdot \overline{p+1}}{r \cdot \overline{r+1}}$, &c. respectively.

The general principles of this case were first delivered by Mr. BERNOULLI, Mr. DE MOIVRE, Mr. EULER, &c.

12. Assume the series $a + bx^n + cx^{2n} +$ &c. $= p$, multiply it into $x^{\alpha-1}\dot{x}$, and find the fluent, then will $\frac{1}{\alpha} x^\alpha p - \frac{1}{\alpha} \int x^\alpha \dot{p} = \frac{1}{\alpha} a x^\alpha +$

$\frac{1}{\alpha+n}bx^{\alpha+n}+\frac{1}{\alpha+2n}cx^{\alpha+2n}+$ &c.; multiply this equation into $x^{\beta-\alpha-1}\dot{x}$, and find the fluent of the equation resulting, which will be $\frac{1}{\beta}\times\frac{1}{\alpha}x^{\beta}p-\frac{1}{\alpha}\cdot\frac{1}{\beta}\int x^{\beta}\dot{p}-\frac{1}{\alpha}\times\frac{1}{\beta-\alpha}x^{\beta-\alpha}\int x^{\alpha}\dot{p}+\frac{1}{\alpha}\cdot\frac{1}{\beta-\alpha}\int x^{\beta}\dot{p}=\frac{1}{\alpha}\cdot\frac{1}{\beta}ax^{\beta}+\frac{1}{\alpha+n}\cdot\frac{1}{\beta+n}bx^{\beta+n}+\frac{1}{\alpha+2n}\cdot\frac{1}{\beta+2n}cx^{\beta+2n}+$ &c.; divide by x^{β}, and there results $\frac{1}{\beta}\cdot\frac{1}{\alpha}p+\frac{1}{\alpha}\cdot\frac{1}{\alpha-\beta}x^{-\alpha}\int x^{\alpha}\dot{p}+\frac{1}{\beta}\cdot\frac{1}{\beta-\alpha}x^{-\beta}\int x^{\beta}\dot{p}=\frac{1}{\alpha}\cdot\frac{1}{\beta}a+\frac{1}{\alpha+n}\cdot\frac{1}{\beta+n}bx^{n}+$ &c.: and in general $\frac{1}{\alpha}\cdot\frac{1}{\beta}\cdot\frac{1}{\gamma}\cdot$ &c. $p+\frac{1}{\alpha}\cdot\frac{1}{\alpha-\beta}\cdot\frac{1}{\alpha-\gamma}x^{-\alpha}\int x^{\alpha}\dot{p}+\frac{1}{\beta}\cdot\frac{1}{\beta-\alpha}\cdot\frac{1}{\beta-\gamma}\cdot$ &c. $x^{-\beta}\int x^{\beta}\dot{p}+\frac{1}{\gamma}\cdot\frac{1}{\gamma-\alpha}\cdot\frac{1}{\gamma-\beta}$, &c. $x^{-\gamma}\int x^{\gamma}\dot{p}=\frac{1}{\alpha}\cdot\frac{1}{\beta}\cdot\frac{1}{\gamma}$ &c. $a+\frac{1}{\alpha+n}\cdot\frac{1}{\beta+n}\cdot\frac{1}{\gamma+n}$ &c. $bx^{n}+\frac{1}{\alpha+2n}\cdot\frac{1}{\beta+2n}\cdot\frac{1}{\gamma+2n}$, &c. $cx^{2n}+$ &c. whence the law of continuation is immediately manifest.

Hence, if no two quantities α, β, γ, δ, &c. be equal to each other; and the successive terms a, b, c, d, &c. of any series $a+bx^{n}+cx^{2n}+$ &c. $=p$ be divided by $\alpha.\beta.\gamma.\delta.$ &c.; $\overline{\alpha+n}\,.\,\overline{\beta+n}\,.\,\overline{\gamma+n}\,.\,\overline{\delta+n}$. &c.; $\alpha+2n\,.\,\beta+2n\,.\,\gamma+2n\,.\,\delta+2n$. &c. &c.; and in general by $\alpha+nz\,.\,\beta+nz\,.\,\gamma+nz\,.\,\delta+nz$. &c. &c.; then can the sum of the series be found from the fluents of the fluxions $x^{\alpha}\dot{p}$, $x^{\beta}\dot{p}$, $x^{\gamma}\dot{p}$, $x^{\delta}\dot{p}$, &c. as has been observed in the Meditationes. If two are equal, *viz.* $\alpha=\beta$, then also the fluent of the fluxion $\frac{\dot{x}}{x}\int x^{\alpha}\dot{p}$ is required. If three are equal *viz.* $\alpha=\beta=\gamma$; then it is necessary to find the fluent of the fluxion $\frac{\dot{x}}{x}\int\frac{\dot{x}}{x}\int x^{\alpha}\dot{p}$; and so on.

1. Let $\dot{p}=\frac{\dot{x}}{1\pm x^{n}}$; and if the differences of the quantities α, β, γ, δ, &c. are divisible by n, from the fluent of the fluxion

fluxion $x^{\alpha}\dot{p}$ can be deduced the fluents of all the other fluxions $x^{\beta}\dot{p}$, $x^{\gamma}\dot{p}$, &c.; and in general, if $\alpha - \beta$ is diviſible by n, then from the fluent of the fluxion $x^{\alpha}\dot{p}$ can be deduced the fluent of the fluxion $x^{\beta}\dot{p}$.

2. Suppoſe $p =$ the terms of the binomial theorem expanded according to the dimenſions of x, *viz.* $(a + bx^{n})^{\frac{r}{s}} = a^{\frac{r}{s}} + \frac{r}{s} a^{\frac{r}{s}-1} bx^{n} +$ &c. beginning from the firſt or any other terms; then, if α, β, &c. divided by n give whole affirmative numbers, will all the fluxions $x^{\alpha}\dot{p}$, $x^{\beta}\dot{p}$, $x^{\gamma}\dot{p}$, &c. be integrable; and if the differences of the quantities α, β, γ, δ, &c. are diviſible by n, from the fluent of the fluxion $x^{\alpha}\dot{p}$ can be deduced the fluents of the fluxions $x^{\beta}\dot{p}$, $x^{\gamma}\dot{p}$, &c.

If p denotes the ſum of the alternate or terms whoſe diſtance from each other are m, of the binomial theorem, the ſame may be applied.

3. If $p = \overline{a + bx^{n} + cx^{2n}}^{\frac{r}{s}}$; and α, β, γ, δ, &c. divided by n give whole affirmative numbers, then from $\int x^{\alpha}\dot{p}$ can be deduced all the remainder $\int x^{\beta}\dot{p}$, $\int x^{\gamma}\dot{p}$, &c.: and in general from two can be deduced all the remainder.

To find when the ſum of any ſeries of this kind can be found, add together each of the fluents, which can be found from each other, and not otherwiſe, and ſuppoſe their ſum $= 0$; and ſo of any other ſimilar fluent, and from the reſulting equations can be diſcovered when the ſeries can be integrated.

13. If the general term of a ſeries contains in it more variable quantities, z, v, w, &c.; then find the ſum of the ſeries, firſt, from the hypotheſis that one of them (z) is only variable,

ble, which, properly correɛted, let be A; in the quantity A ſuppoſe all the quantities invariable but ſome other v, and find the ſum of the ſeries thence reſulting, which let be B, and ſo on; and the ſum of the ſeries will be deduced.

Ex. Let the term be $\frac{1}{z \cdot \overline{z+1} \times v \cdot \overline{v+1} \cdot \overline{v+2}}$; the dimenſions of z and v, &c. in the denominator muſt be at leaſt greater than its dimenſions in the numerator by a quantity greater than the number of the quantities z, v, &c. which proceed *in infinitum* increaſed by unity. Firſt, ſuppoſe z only variable, and the ſum of the infinite ſeries reſulting will be $\frac{1}{z \cdot v \cdot \overline{v+1} \cdot \overline{v+2}} = A$; then ſuppoſe v only variable, and the ſum reſulting will be $\frac{1}{2z \cdot v \cdot \overline{v+1}} = B$, which is the ſum required.

If it be ſuppoſed, that the quantities z and v, &c. in the ſame term ſhall never have the ſame values, then ſuppoſe z and v always to have the ſame values, and the general term $\frac{1}{z \cdot \overline{z+1} \cdot v \cdot \overline{v+1} \cdot \overline{v+2}}$ becomes $\frac{1}{z^2 \cdot \overline{z+1}^2 \cdot \overline{z+2}}$, of which let the ſum be V, then will B − V be the ſum required.

On this and ſome other ſubjeɛts more have been given in the Meditationes.

14. If the ſum of the ſeries cannot be found in finite terms, and it is neceſſary to recur to infinite ſeries; it is obſerved in the Meditationes to be generally neceſſary to add ſo many terms together, that the diſtance from the firſt term of the ſeries may conſiderably exceed the greateſt root of an equation reſulting from the general term made $= 0$; and afterwards a ſeries more converging may commonly be deduced from the fluents of fluxions reſulting from negleɛting all but the greateſt quantities in the general terms reſulting; and by other

different methods. Mr. NICHOLAS BERNOULLI and Mr. MOMMORT investigated the sum of the series (P) $A + Br + Cr^2 +$ &c. by a series (Q) $\frac{A}{1-r} + \frac{d'r}{(1-r)^2} + \frac{d''r^2}{(1-r)^3} + \frac{d'''r^3}{(1-r)^4} +$ &c.; where d', d'', d''', &c. denote the successive differences of the terms A, B, C, D, &c. If r be negative, the denominators become $1+r$, $(1+r)^2$, $(1+r)^3$, &c.

It has been observed, in the Meditationes, that in swift converging series the series P will converge more swiftly than the series Q; in series converging according to a geometrical ratio, sometimes the one will converge more swift, and sometimes the other. In other series, which converge more slow, where most commonly r nearly $= 1$, it cannot in general be said, which of the serieses will converge the swiftest. The preceding remark, *viz.* the addition of the first terms of the series, is necessary in most cases of finding the sums by serieses of this kind.

It is not unworthy of observation, that in almost all cases of infinite series, the convergency depends on the roots of the given equations, which remark was first published in the Meditationes. For example: in finding approximates to the roots of given equations the convergency depends on how much the approximates given are more near to one root than to any other; and consequently, when two or more roots or values of an unknown quantity are nearly equal, different rules are to be applied, which are improvements of the rule of false. This rule, and the above-mentioned observations were first given in the Meditationes Algebraicæ et Analyticæ, with several other additions on similar subjects.

Many

Many more things concerning the ſummation of ſeries, which depend on fluxional, &c. equations, might be added; but I ſhall conclude this paper with congratulating myſelf, that ſome algebraical inventions publiſhed by me have been ſince thought not unworthy of being publiſhed by ſome of the greateſt mathematicians of this or any other age.

1ſt, In the year 1757, I ſent to the Royal Society the firſt edition of my Meditationes Algebraicæ: they were printed and publiſhed in the years 1760 and 1762, with Properties of Curve Lines, under the title of Miſcellanea Analytica, and a copy of them ſent to Mr. EULER in the beginning of the year 1763, in which was contained a reſolution of algebraical equations, not inferior, on account of its generality and facility, to any yet publiſhed (*viz.* $y = a\sqrt[n]{p} + b\sqrt[n]{p^2} + c\sqrt[n]{p^3} + \ldots \sqrt[n]{p^{n-1}}$). This reſolution was publiſhed by Mr. EULER in the Peterſburg Acts for the year 1764. Whether Mr. EULER ever received my book, I cannot pretend to ſay; nor is it material: for the fact is, that it was publiſhed by me in the year 1760 and 1762, and firſt by Mr. EULER in the year 1764. Mr. DE LA GRANGE and Mr. BEZOUT have aſcribed this reſolution to Mr. EULER, as firſt publiſhed in the year 1764, not having ſeen (I ſuppoſe) my Miſcell. Analyt. Mr. BEZOUT found from it ſome new equations, of which the reſolution is known, and applied it to the reduction of equations: more new equations are given, and the reſolution rendered more eaſy by me in the Philoſophical Tranſactions. 2d, In the above-mentioned Miſcell. Analyt. an equation is transformed into another, of which the roots are the ſquares of the differences of the roots of the given equation; and it is aſſerted in that book, that if the coefficients co-efficients of the terms of the refulting equations change continually from + to – and – to +, the roots of the given equation are all poffible, otherwife not; and in a paper, inferted by me in the Philofophical Tranfactions for the year 1764, in which is found from this transformation, when there are none, two or four impoffible roots contained in an algebraical equation of four or five dimenfions; it is obferved, that there will be none or four, &c. impoffible roots contained in the given equation, if the laft term be + or –; and two, &c. on the contrary, if the laft term be – or +. Thefe obfervations and tranfformation have been fince publifhed and explained in the Berlin Acts for the years 1767 and 1768, by Mr. DE LA GRANGE. 3d. In the Mifcell. Anal. an equation is transformed into another, whofe roots are the fquares, &c. of the roots of a given equation; and it is afferted, that there are at leaft fo many impoffible roots contained in the given equation, as there are continual progreffes in the refulting equation from + to + and – to –. It is afterwards remarked, that thefe rules fometimes find impoffible roots when Sir ISAAC NEWTON's, and fuch like rules, fail; and that Sir ISAAC NEWTON's, &c. will find them, when this rule fails. This rule may fomewhat further be promoted by firft changing the given equation, whofe root is x, into another whofe root is $\sqrt{-1}x$; but, in my opinion, the rule of HARRIOT's, which only finds whether there are impoffible roots contained in a cubic equation or not, is to be preferred to thefe rules, which, in equations of any dimenfions, of which the impoffible roots cannot generally be found from the rules, feldom find the true number. 4th, It is remarked, that rules which difcover the true number of impoffible roots require immenfe calculations, fince they muft neceffarily find, when

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the roots become equal. In order to this, in the Miſcell. Anal. there is found an equation, whoſe roots are the reciprocals of the differences of any two roots of the given equation; and from finding a quantity (π) greater than the greateſt root of the given, and $\left(\frac{1}{A}\right)$ greater than the greateſt root of the reſulting equation, and ſubſtituting π, $\pi - A$, $\pi - 2A$, &c. for x in the given equation; will always be found the true number of impoſſible roots. 5th, In the ſame book are aſſumed two equations ($nx^{n-1} - \overline{n-1}px^{n-2} + \overline{n-2}qx^{n-3} - \text{\&c.} = 0$ and $x^n - px^{n-1} + \text{\&c.} = w$), and thence deduced an equation, whoſe root is w, from which, in ſome caſes, can be found the number of impoſſible roots.

6. In the Miſcell. Anal. is given the law of a ſeries, and its demonſtration, which finds the ſum of the powers of the roots of a given equation from its co-efficients. Mr. EULER has ſince publiſhed the ſame in the Peterſburg Acts. Mr. DE LA GRANGE printed a property of this ſeries, alſo printed by me about the ſame time; *viz.* that if the ſeries was continued *in infinitum*, the powers would obſerve the ſame law as the roots, which indeed immediately follows from the ſeries itſelf; but from thence with the greateſt ſagacity he deduces the law of the reverſion of the ſeries ($y = a + bx + cx^2 + dx^3 + \text{\&c.}$): it has ſince been given in a different manner from ſimilar principles in the Medit. Analyt. 7. In the Miſcell. Analyt. the law of a ſeries is given for finding the ſum of all quantities of this kind ($\alpha^m \times \beta^n \times \gamma^r \times \delta^s \times \text{\&c.} + \text{\&c.}$) where α, β, γ, δ, &c. denote the roots of a given equation, from the powers of the roots of the given equation. This law, with a different notation, has been ſince publiſhed in the Paris Acts by Mr. VANDERMONDE; who indeed mentions that he

had heard, that a ſeries for that purpoſe was contained in my book, but had not ſeen it. In the ſame book is given a method of finding the aggregates of any algebraical functions of each of the roots of given equations, which is ſomewhat improved in the latter editions. 8. In the ſame book are aſſumed $\frac{az^n+bz^{n-1}+\&c.}{pz^m+qz^{m-1}+\&c.}$ and $\frac{Az^{n'}+Bz^{n'-1}+\&c.}{pz^m+qz^{m-1}+\&c.}$, where z is any rational quantity whatever for x and y, the unknown quantities of a given equation of two or more dimenſions. 9. In the Miſcell. Analyt. a biquadratic ($x^4+2px^3=qx^2+rx+s$, of which no term is deſtroyed) is reduced to a quadratic ($x^2+px+n=\sqrt{p^2+2n+q}\,x+\sqrt{s+n^2}$;) and in the ſecond edition of it, printed in the years 1767, 1768, 1769, and publiſhed in the beginning of the year 1770, the values of n are found $\frac{\alpha\beta+\gamma\delta}{2}$, $\frac{\alpha\gamma+\beta\delta}{2}$, and $\frac{\alpha\delta+\beta\gamma}{2}$; and the ſix values of $\sqrt{y^2+2n+q}$ reſpectively $\frac{\alpha+\beta-\gamma-\delta}{2}$, $\frac{\alpha+\gamma-\beta-\delta}{2}$, $\frac{\alpha+\delta-\beta-\gamma}{2}$, and their negatives; and the ſix values of $\sqrt{s+n^2}$ reſpectively $\frac{\alpha\beta-\gamma\delta}{2}$, $\frac{\alpha\gamma-\beta\delta}{2}$, $\frac{\alpha\delta-\beta\gamma}{2}$, and their negatives. 10. From a given biquadratic ($y^4+qy^2+ry+s=0$) by aſſuming $y^2+ay+b=v$ and a and b ſuch quantities as to make the ſecond and fourth terms of the reſulting equations to vaniſh, there reſults an equation ($v^4+Av^2+B=0$) of the formula of a quadratic. Mr. DE LA GRANGE has aſcribed this reſolution to Mr. TSCHIRNHAUSEN; but in the Leipſic Acts the reſolution of a cubic is given by Mr. TSCHIRNHAUSEN, but not of a biquadratic: his general deſign ſeems to be the extermination of all the terms.

 11. Mr.

11. Mr. EULER or Mr. DE LA GRANGE found, that if α be a root of the equation $x^n - 1 = 0$, where n is a prime number, α, α^2, α^3, . . α^{n-1}, 1 will be (n) roots of it. More on a similar subject has been added in the last edition of the Medit. Algebr. 12. It is observed in the Miscell. Analyt. that CARDAN's or SCIPIO FERREUS's resolution of a cubic is a resolution of three different cubic equations; and in the Medit. Algeb. 1770, the three cubics are given, and the rationale of the resolution (for example: if α, β, and γ, be the roots of the cubic equation $x^3 + qx - r = 0$, then is given the function of the above roots, which are the roots of the reducing equation $z^6 - rz^3 = q^3$); and also the rationale of the common resolution of biquadratics. 13. It is asserted in the Miscell. that if the terms ($My^n + by^{n-1}x + cy^{n-2}x^2 +$ &c. and $Ny^m + By^{m-1}x + Cy^{m-2}x^2 +$ &c.) of two equations of n and m dimensions, which contain the greatest dimensions of x and y have a common divisor, the equation whose root is x or y, will not ascend to $n \times m$ dimensions; and if the equation, whose root is x or y, ascends to $n \times m$ dimensions, the sum of its roots depends on the terms of n and $n - 1$ dimensions in the one, and m and $m - 1$ dimensions in the other equation, &c. It is also asserted, in the Miscell. that if three algebraical equations of n, m, and r dimensions contain three unknown quantities x, y, and z, the equation, whose root is x or y or z, cannot ascend to more than $n . m . r$ dimensions. 14. Mr. BEZOUT has given two very elegant propositions for finding the dimensions of the equation whose root is x or y, &c ; where x, y, &c. are unknown quantities contained in two or more (b) algebraical equations of π, ρ, σ, &c. dimensions, and in which some of the unknown quantities do not ascend to the above π, ρ, σ, &c. dimensions

dimenſions reſpectively. In demonſtrating theſe propoſitions he uſes one (amongſt others) before given by me (*viz.* if an equation of n dimenſions contains m unknown quantities, the number of different terms which may be contained in it will be $\overline{n+1} \cdot \frac{n+2}{2} \cdot \frac{n+3}{3} \cdot\cdot \frac{n+m}{m}$). In the Medit. 1770 there is given a method of finding in many caſes the dimenſions of the equation, whoſe root is x or y, &c.; from which one, if not both, of the above-mentioned caſes may more eaſily be deduced, and others added. 15. In the Medit. 1770 is obſerved, that if there be n equations containing m unknown quantities, where n is greater than m, there will be $n-m$ equations of conditions, &c. 16. In the Miſcell. is given and demonſtrated the ſubſequent propoſition; *viz.* if two equations contain two unknown quantities x and y, in which x and y are ſimilarly involved; the equation, whoſe root is x or y will have twice the number of roots which the equation, whoſe root is $x+y$, x^2+y^2, &c. has. In the Medit. 1770 the ſame reaſoning is applied to equations, which have two, three, four, &c. quantities ſimilarly involved. 17. Mr. DE LA GRANGE has done me the honour to demonſtrate my method of finding the number of affirmative and negative roots contained in a biquadratic equation. A demonſtration of my rule for finding the number of affirmative, negative, and impoſſible roots contained in the equation $x^n + Ax^m + B = 0$ is alſo omitted, on account of its eaſe and length. From the Medit. the inveſtigation of finding the true number of affirmative and negative roots appears to be as difficult a problem as the finding the true number of impoſſible roots; and it further appears, that the common methods in both caſes can ſeldom be depended on. But their faults lie on different ſides, the one generally finds too many, the other too few. 18. In the Medit. 1770, from the number of impoffible roots in a given equation ($x^n - px^{n-1} + \&c. = 0$) is found the number of impoffible roots in an equation, whofe roots (v) have any affignable relation to the roots of a given equation; and examples are given in the relation ($nx^{n-1} - \overline{n-1}px^{n-2} + \&c. = v$); and in an equation, whofe roots are the fquares of the differences of the roots of the given equation. 19. It is obferved in the Medit. 1770, that in two or more equations, having two or more unknown quantities, the fame irrationality will be contained in the correfpondent values of each of the unknown quantities, unlefs two or more values of one of them are equal, &c. The fame obfervation is alfo applied to the coefficients of an equation deduced from a given equation. 20. In the Mifcell. was publifhed a new method of exterminating, from a given equation, irrational quantities, by finding the the multipliers, which, multiplied into it, give a rational product. 21. In the Medit. 1770, are given the different refolutions of a certain quantity $(a^2 + rb^2)^{2m+1}$ and $(a^2 + rb^2)^{2m+2}$ into quantities of the fame kind. 22. Mr. DE LA GRANGE has very elegantly demonftrated Mr. WILSON's celebrated property of prime numbers contained in my book. In the laft edition of the Medit. the fame property is demonftrated, and fome fimilar ones added. 23. In the Mifcell. is given a method of finding all the integral correfpondent values of the unknown quantities of a given fimple equation, having two or more unknown quantities; and, in the Medit. 1770, are given methods of reducing fimple and other algebraical equations into one, fo that fome unknown quantities may be exterminated; and if the unknown quantities of the refulting equations be integral or rational,

rational, the unknown quantities exterminated may alſo be integral or rational. 24. In the Medit. are given rules for finding the different and correſpondent roots of an equation, whoſe reſolution is given. 25. Mr. DE LA GRANGE has recommended my new transformation of equations, publiſhed in the Miſcell. which perhaps is not leſs general nor elegant than any yet publiſhed; and in the Meditat. 1770 is given a method very uſeful in finding the co-efficients.

If either here, or in the preface to the Medit. Algebraicæ, I have aſcribed to myſelf any algebraical, or in the properties of curve lines any geometrical, or in the Medit. Analyt. any analytical invention, which has been before publiſhed by any other perſon, I can only plead ignorance of it, and ſhall on the very firſt conviction acknowledge it.

I muſt further add, that I have been able to carry my algebraical improvements into geometry; for from them, with ſome geometrical principles added, I have (unleſs I am deceived) deduced as many new properties of conic ſections and curve lines as have been publiſhed by any one ſince the great geometrician APOLLONIUS.